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Abstract

We represent the ensemble of extremal copositive matrices $A \in \mathcal{COP}^n$ of the copositive cone as a disjoint union of a finite number of algebraic manifolds. These manifolds are strata of algebraic sets, each of which is characterized by a finite collection $\mathcal{E}$ of pairs of index sets. We give necessary conditions on $\mathcal{E}$ to correspond to an algebraic set containing exceptional extremal matrices of $\mathcal{COP}^n$. A manifold of extremal matrices is essential if it is not contained in the closure of another such manifold. We compute the essential manifolds of extremal matrices of the cone $\mathcal{COP}^6$.

Keywords: copositive matrix, minimal zero, extreme ray, algebraic manifolds
MSC: 15B48; 90C26

1 Introduction

An element A of the space $\mathcal{S}^n$ of real symmetric $n \times n$ matrices is called *copositive* if $x^T Ax \geq 0$ for all vectors $x \in \mathbb{R}^n_+$. The set of such matrices forms the *copositive cone* $\mathcal{COP}^n$. This cone plays an important role in non-convex optimization, as many difficult optimization problems can be reformulated as conic programs over $\mathcal{COP}^n$. For a detailed survey of the applications of this cone see, e.g., [6, 2, 3, 9].

An important characteristic of the copositive cone in relation to optimization over $\mathcal{COP}^n$ are its extreme rays. Knowledge of the extreme rays allows, e.g., to check the exactness of tractable inner relaxations of the cone [5]. In this contribution we investigate the structure of the ensemble of extreme rays of $\mathcal{COP}^n$. The main tool will be the decomposition of $\mathcal{COP}^n$ into a finite union of subsets $S_{\mathcal{E}}$ of algebraic sets $Z_{\mathcal{E}}$, where $S_{\mathcal{E}}$ is the set of copositive matrices having extended minimal zero support set $\mathcal{E}$ [8]. Here the extended minimal zero support set $\operatorname{esupp} \mathcal{V}^A_{\min}$ of a copositive matrix A is a finer characteristic than the minimal zero support set $\operatorname{supp} \mathcal{V}^A_{\min}$, which was used in [1] to classify the extreme rays of $\mathcal{COP}^6$.

On each irreducible component of $Z_{\mathcal{E}}$, the set $S_{\mathcal{E}}$ either does not contain extremal elements of $\mathcal{COP}^n$ at all, or it consists entirely of extremal elements with the possible exception of an algebraic subset of lower dimension. In addition to the necessary conditions derived in [7] on the minimal zero support set of an exceptional extreme copositive matrix, in Section 2 we shall present necessary conditions on $\mathcal{E}$ for $S_{\mathcal{E}}$ to contain exceptional extremal matrices.

Often it is sufficient to consider a dense subset of the set of extremal elements. For instance, if a closed convex inner relaxation of the copositive cone contains such a subset, then it is exact. In this context it is helpful to introduce the concept of *essential* and *non-essential* manifolds of extremal elements. Here such a manifold is called *essential* if it is not contained in the closure of another manifold of extremal elements [1]. In Section 3 we compute the essential manifolds of extremal elements of the cone $\mathcal{COP}^6$.

1.1 Notations and preliminaries

The space of real symmetric matrices of size $n \times n$ will be denoted by $\mathcal{S}^n$, the cone of element-wise nonnegative matrices by $\mathcal{N}^n$.

For an index set $I \subset \{1, \dots, n\}$, denote by $\overline{I}$ its complement $\{1, \dots, n\} \setminus I$.

We shall denote vectors with lower-case letters and matrices with upper-case letters. Individual entries of a vector u and a matrix A will be denoted by u_i and A_{ij} respectively. For a matrix A and a vector u of compatible dimension, the i-th element of the matrix-vector product Au will be denoted by $(Au)_i$. Inequalities $u \geq \mathbf{0}$ on vectors will be meant element-wise, where we denote by $\mathbf{0} = (0, \dots, 0)^T$ the all-zeros vector. Similarly we denote by $\mathbf{1} = (1, \dots, 1)^T$ the all-ones vector. We further let e_i be the unit vector with i-th entry equal to one and all other entries equal to zero. For a subset $I \subset \{1, \dots, n\}$ we denote by A_I the principal submatrix of A whose elements have row and column indices in I, i.e. $A_I = (A_{ij})_{i,j \in I} \in \mathcal{S}^{|I|}$. For subsets $I, J \subset \{1, \dots, n\}$ we denote by $A_{I \times J}$ the submatrix of A whose elements have row indices in I and column indices in J. Similarly for a vector $u \in \mathbb{R}^n$ we define the subvector $u_I = (u_i)_{i \in I} \in \mathbb{R}^{|I|}$. By E_{ij} we denote a matrix which has all entries equal to zero except (i, j) and (j, i), which equal 1.

Let $\Delta = \{u \in \mathbb{R}^n_+ \mid \mathbf{1}^T u = 1\}$ be the standard simplex.

For a nonnegative vector $u \in \mathbb{R}^n_+$ we define its *support* as $\operatorname{supp} u = \{i \in \{1, \dots, n\} \mid u_i > 0\}$.

A zero u of a copositive matrix A is called *minimal* if there exists no zero v of A such that the inclusion $\operatorname{supp} v \subset \operatorname{supp} u$ holds strictly. We shall denote the set of minimal zeros of a copositive matrix A by $\mathcal{V}^A_{\min}$ and the ensemble of supports of the minimal zeros of A by $\operatorname{supp} \mathcal{V}^A_{\min}$. To each index set I there exists at most one minimal zero $u \in \Delta$ of A with $\operatorname{supp} u = I$ [7, Lemma 3.5], hence the minimal zero support set $\operatorname{supp} \mathcal{V}^A_{\min}$ is in bijective correspondence to the minimal zeros of A which are contained in Δ.

For a zero u of a copositive matrix A, the matrix-vector product Au is nonnegative. We call the set $\operatorname{comp} u := \overline{\operatorname{supp}(Au)}$ the *complementary index set* of the zero, and the pair $\operatorname{esupp} u = (\operatorname{supp} u, \operatorname{comp} u)$ of index sets the *extended support*. The ensemble of extended supports of the minimal zeros of A will be called the *extended minimal zero support set* and denoted by $\operatorname{esupp} \mathcal{V}^A_{\min}$.

A non-zero copositive matrix $A \in \mathcal{COP}^n$ is called *extremal* if whenever $A = A_1 + A_2$ with $A_1, A_2 \in \mathcal{COP}^n$, the summands A_1, A_2 must be nonnegative multiples of A.

A copositive matrix A is called *irreducible* with respect to another copositive matrix C if for every $\delta > 0$, we have $A - \delta C \notin \mathcal{COP}^n$, and it is called irreducible with respect to a subset $\mathcal{M} \subset \mathcal{COP}^n$ if it is irreducible with respect to all nonzero elements $C \in \mathcal{M}$.

By [4, Lemma 2.5] we have that $\operatorname{supp} u \subset \operatorname{comp} u$ for every zero u of a copositive matrix A.

We now briefly recollect the necessary results from [8]. Let $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$ be a collection of pairs of index sets. Define the sets

$$S_{\mathcal{E}} = \{A \in \mathcal{COP}^n \mid \operatorname{esupp} \mathcal{V}^A_{\min} = \mathcal{E}\}, \qquad Z_{\mathcal{E}} = \{A \in \mathcal{S}^n \mid A_{I_\alpha, J_\alpha} \text{ is rank deficient } \forall\ \alpha = 1, \dots, m\}.$$

The set $Z_{\mathcal{E}}$ is algebraic, given by the zero locus of a finite number of determinantal polynomials. The set $S_{\mathcal{E}}$ is a relatively open subset of $Z_{\mathcal{E}}$ [8, Corollary 1].

The set $S_{\mathcal{E}}$ is a disjoint union of interiors of faces of $\mathcal{COP}^n$ [8, Corollary 2]. Moreover, if C is an irreducible component of $Z_{\mathcal{E}}$, then the dimension of the faces over $S = C \cap S_{\mathcal{E}}$ is generically constant. On an algebraic subset the faces may have a higher dimension [8, Lemma 7].

2 Structure of the set of extremal matrices

In this section we establish some properties of the extended minimal zero support set and the corresponding sets $S_{\mathcal{E}}$ in relation to the extremality of matrices in $\mathcal{COP}^n$.

Lemma 2.1. *Let $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$ be an arbitrary collection of pairs of index sets, and let C be an irreducible component of the algebraic set $Z_{\mathcal{E}}$. Then either $S = C \cap S_{\mathcal{E}}$ contains no extremal matrix, or all matrices in S are extremal with the possible exception of an algebraic subset of lower dimension.*

Proof. An extremal matrix is characterized by the condition that the dimension of its minimal face in $\mathcal{COP}^n$ equals 1. The assertion then follows directly from [8, Lemma 7]. □

Since C is an algebraic variety and hence a stratified algebraic manifold [11, 10], we have that the set of extremal matrices in S is a finite union of algebraic manifolds. We shall call these manifolds *components* of the set of extremal matrices of $\mathcal{COP}^n$. We may then introduce the following notion [1].

Definition 2.2. A component of extremal matrices of $\mathcal{COP}^n$ is called *essential* if it is not contained in the closure of another such component.

From [8, Lemma 4] we have the following result.

Lemma 2.3. *Let a component C' of extremal matrices be contained in the closure of another component C. Suppose further $C \subset S_{\mathcal{E}}$, $C' \subset S_{\mathcal{E}'}$, where $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$, $\mathcal{E}' = \{(I'_\alpha, J'_\alpha)\}_{\alpha=1,\dots,m'}$. Then for every $\alpha = 1,\dots,m$ there exists $\alpha' \in \{1,\dots,m'\}$ such that $I'_{\alpha'} \subset I_\alpha$, $J_\alpha \subset J'_{\alpha'}$.* □

An even stronger relation than that in Lemma 2.1 can be established for the condition of irreducibility with respect to the matrix E_{ij} and the cone $\mathcal{N}^n$. Indeed, [7, Lemma 4.1] and [7, Corollary 4.2] can be equivalently rewritten in terms of the extended minimal zero support set.

Lemma 2.4. *Let $A \in \mathcal{COP}^n$ and let $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$ be the extended minimal zero support set of A. Then A is irreducible with respect to E_{ij} if and only if there exists α such that $(i,j) \in (I_\alpha \times J_\alpha) \cup (J_\alpha \times I_\alpha)$. The matrix A is irreducible with respect to $\mathcal{N}^n$ if and only if $\bigcup_{\alpha=1}^m (I_\alpha \times J_\alpha) \cup (J_\alpha \times I_\alpha) = \{1,\dots,n\}^2$.* □

This yields the following corollary.

Corollary 2.5. *Let (i,j) be an index pair, and $\mathcal{E}$ a collection of pairs of index sets. Then either all matrices or no matrix in the subset $S_{\mathcal{E}}$ is irreducible with respect to E_{ij} (respectively with respect to the cone $\mathcal{N}^n$).*

Proof. By Lemma 2.4 the irreducibility of a matrix A with respect to E_{ij} or $\mathcal{N}^n$ depends on the extended minimal zero support set $\mathcal{E}$ of A only. □

We have also the following consequence.

Corollary 2.6. *Let a collection $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$ of pairs of index sets be such $S_{\mathcal{E}}$ contains exceptional extremal matrices. Then $\mathcal{E}$ satisfies the condition $\bigcup_{\alpha=1}^m (I_\alpha \times J_\alpha) \cup (J_\alpha \times I_\alpha) = \{1,\dots,n\}^2$.*

Proof. The corollary follows from Lemma 2.4 and the necessity for an exceptional extremal matrix to be irreducible with respect to $\mathcal{N}^n$. □

There are also other necessary conditions on the collection $\mathcal{E}$.

Lemma 2.7. *Let $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$ be a collection of pairs of index sets such that $S_{\mathcal{E}}$ contains exceptional extremal matrices. Then for every α, β we have $I_\alpha \subset J_\beta$ if and only if $I_\beta \subset J_\alpha$.*

Proof. The assertion follows directly from [8, Lemma 3]. □

In [7] another collection of necessary conditions on the minimal zero supports I_α alone has been established.

Similar to the minimal zero support set, the extended minimal zero support set can hence be used for the classification of the extreme rays of the cone $\mathcal{COP}^n$.

3 Essential components of extremal matrices in $\mathcal{COP}^6$

In this section we compute the essential components of exceptional extremal matrices of the cone $\mathcal{COP}^6$. To this end we first collect all components of exceptional extremal matrices of $\mathcal{COP}^6$ from [1] in Table 1. Some of them are contained in the closure of other components already by construction, as shown in the last column of Table 1. We then use the criterion provided by Lemma 2.3 to establish which further components could possibly be in the closure of which other components. For each component, we then either establish or refute its being non-essential by direct verification.

The minimal zero support sets of exceptional extremal matrices in $\mathcal{COP}^6$ are taken from [1, Table 1]. However, matrices having the same minimal zero support set can belong to different sets $S_{\mathcal{E}}$, i.e., have different extended minimal zero support sets. For each such sub-case we provide in Table 1 the complementary index set J_α for each of the minimal zeros u^α, but for brevity we present only the differences $J_\alpha \setminus I_\alpha$, where I_α is the support of u^α.

Note that some of the cases from [1] are split into several cases in Table 1, according to whether one or more non-strict inequalities on the angle parameters are equalities. An inequality becoming an equality is accompanied by the appearance of additional indices in the complementary support of one or more minimal zeros. As a consequence, the corresponding boundary piece belongs to a different set $S_{\mathcal{E}}$ and is listed as a separate component in Table 1.

Note that in Case 15 the two non-strict inequalities on the angle parameters are equivalent under the permutation (124365) of the indices $\{1, \dots, 6\}$, and cannot both be equalities. Hence this case gives rise to two non-equivalent components only.

In Table 1 we provide also the dimension of each component, taken from [1, Table 2].

It turns out that in the cone $\mathcal{COP}^6$ different components of exceptional extremal copositive matrices correspond to different extended minimal zero support sets $\mathcal{E}$, i.e., each $S_{\mathcal{E}}$ consists only of one component.

From Table 1 it is apparent that only 22 of the components may possibly be essential. For every ordered pair in this list we check whether the necessary criterion in Lemma 2.3 for the first component being in the closure of the second is satisfied. Another necessary criterion is that the dimension of the first component is strictly smaller than the dimension of the second one.

Note that Table 1 lists the extended minimal zero support sets only up to a permutation of the index set $\{1, \dots, 6\}$. We hence have to allow for this freedom when checking the criterion in Lemma 2.3 on pairs of extended minimal zero supports.

The results of this test are collected in Table 2.

From Table 2 it follows that the components 13.1,13.2,16.1,17,19.1 are essential.

For the remaining 17 components we now show that they lie on the boundary of other components and are hence non-essential, by providing an appropriate permutation of the indices $\{1, \dots, 6\}$ and an explicit limit.

- **Case O5.2** is in the closure of **Case 18** when the last row and column tend to zero.
- **Case 14** is in the closure of **Case 1** after the substitution $\phi_1 \to 0$, $\phi_2 \to 0$.
- **Case 1** is in the closure of **Case 2** after the substitution $\phi_1 \to \phi_2$, $\phi_2 \to 0$, $\phi_3 \to \pi - \phi_1 - \phi_2$.
- **Case 2** is in the closure of **Case 3** after the permutation (124356) and the substitution $\phi_1 \to \phi_1$, $\phi_2 \to \pi - \phi_1$,$\phi_3 \to \phi_3$, $\phi_4 \to \phi_2$.
- **Case 3** is in the closure of **Case 5** after the permutation (152346) and the substitution $\phi_1 \to \phi_1$,$\phi_2 \to (\pi - \phi_1 - \phi_2)$,$\phi_3 \to \phi_4$, $\phi_4 \to \phi_3$,$\phi_5 \to 0$.
- **Case 10** is in the closure of **Case 17** after the permutation (241536) and the substitution $\phi_1 \to \phi_3$, $\phi_2 \to \phi_2$, $\phi_3 \to 0$, $\phi_4 \to \pi - \phi_1 - \phi_2$, $\phi_5 \to \phi_6$, $\phi_6 \to \phi_5$, $\phi_7 \to \phi_4$.
- **Case 11** is in the closure of **Case 19.1** after the permutation (514263) and the substitution $\phi_1 \to \phi_4$, $\phi_2 \to \phi_1$, $\phi_3 \to \phi_2$, $\phi_4 \to \phi_3$, $\phi_5 \to \phi_5$, $\phi_6 \to \pi - \phi_2 - \phi_6$, $\phi_7 \to \pi - \phi_6 + \phi_3$, $a_{24} \to \cos(\phi_4 + \phi_5)$, $a_{36} \to b_3$.
- **Case 12** is in the closure of **Case 19.1** after the permutation (152463) and the substitution $\phi_1 \to \phi_6$, $\phi_2 \to \phi_3$, $\phi_3 \to \phi_2$, $\phi_4 \to \phi_1$, $\phi_5 \to \phi_5$, $\phi_6 \to \phi_4$, $\phi_7 \to \pi - \phi_5 - \phi_7$, $a_{24} \to b_1$, $a_{36} \to -\cos\phi_7$.
- Part $0 \leq \phi_6 < \phi_2$ of case 18 is in the closure of case 16.1 after the permutation (631254) and the substitution $\phi_1 \to \phi_5$, $\phi_2 \to \phi_1$, $\phi_3 \to \phi_2 - \phi_6$, $\phi_4 \to \phi_2$, $\phi_5 \to \phi_4$, $\phi_6 \to \phi_3 + \phi_6$, $\phi_7 \to \phi_3$. Using the permutation (213654) instead we obtain the part $-\phi_3 < \phi_6 \leq 0$ of case 18, because this part is obtained from the former part by the permutation (432156). As a result, **Case 18** is in the closure of **Case 16.1**.
- **Case 15.1** is in the closure of **Case 16.1** after the permutation (654321) and the substitution $\phi_1 \to \phi_4$, $\phi_2 \to \phi_5$, $\phi_3 \to \phi_3$, $\phi_4 \to \pi - \phi_5 - \phi_6$, $\phi_5 \to \phi_1$, $\phi_6 \to \phi_2$, $\phi_7 \to 0$.
- **Case 9.1** is in the closure of **Case 16.1** after the permutation (241356) and the substitution $\phi_1 \to \phi_1$, $\phi_2 \to \phi_2$, $\phi_3 \to 0$, $\phi_4 \to \pi - \phi_2 - \phi_3$, $\phi_5 \to \phi_4$, $\phi_6 \to \phi_5$, $\phi_7 \to \phi_6$.
- **Case 9.2** is in the closure of **Case 16.1** after the permutation (643152) and the substitution $\phi_1 \to \phi_1$, $\phi_2 \to \phi_4$, $\phi_3 \to \phi_5$, $\phi_4 \to \phi_6$, $\phi_5 \to \phi_2$, $\phi_6 \to 0$, $\phi_7 \to \pi - \phi_2 - \phi_3$.
- **Case 8.1** is in the closure of **Case 16.1** after the permutation (316452) and the substitution $\phi_1 \to 0$, $\phi_2 \to \phi_5$, $\phi_3 \to \phi_4$, $\phi_4 \to \phi_6$, $\phi_5 \to \phi_2$, $\phi_6 \to \phi_1$, $\phi_7 \to \phi_3$.

- **Case 5** is in the closure of **Case 16.1** after the permutation (654132) and the substitution $\phi_1 \to \phi_2$, $\phi_2 \to \phi_1$, $\phi_3 \to \phi_3$, $\phi_4 \to \phi_4$, $\phi_5 \to \phi_5$, $\phi_6 \to 0$, $\phi_7 \to 0$.
- **Case 7.1** is in the closure of **Case 16.1** after the permutation (463125) and the substitution $\phi_1 \to \phi_3$, $\phi_2 \to \phi_4$, $\phi_3 \to 0$, $\phi_4 \to \phi_5$, $\phi_5 \to \phi_2$, $\phi_6 \to \phi_1$, $\phi_7 \to 0$.
- **Case 6** is in the closure of **Case 16.1** after the permutation (426135) and the substitution $\phi_1 \to \phi_3$, $\phi_2 \to \phi_2$, $\phi_3 \to 0$, $\phi_4 \to \pi - \phi_2 - \phi_4$, $\phi_5 \to \phi_1$, $\phi_6 \to 0$, $\phi_7 \to \phi_5$.
- **Case 4** is in the closure of **Case 16.1** after the permutation (645213) and the substitution $\phi_1 \to \phi_3$, $\phi_2 \to \phi_2$, $\phi_3 \to 0$, $\phi_4 \to \phi_1$, $\phi_5 \to \phi_4$, $\phi_6 \to 0$, $\phi_7 \to 0$.

We obtain the following result.

Theorem 3.1. *Out of the 36 mutually non-equivalent components of exceptional extremal matrices in $\mathcal{COP}^6$ exactly the 5 components 13.1,13.2,16.1,17,19.1 in Table 1 are essential.* □

Recall that in the case of the cone $\mathcal{COP}^5$, there are two mutually non-equivalent components of exceptional extremal matrices, of which one is essential.

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No.	sets	extended minimal zero support set	dim	in closure of
O5.1	I	{1,2},{2,3},{3,4},{4,5},{1,5},{6}	5	O5.2
	$J \setminus I$	{3,5},{1,4},{2,5},{1,3},{2,4},{1,2,3,4,5}		
O5.2	I	{1,2,3},{2,3,4},{3,4,5},{1,4,5},{1,2,5},{6}	10	
	$J \setminus I$	∅,∅,∅,∅,∅,{1,2,3,4,5}		
1	I	{1,2},{1,3},{1,4},{2,5},{3,6},{4,5,6}	8	
	$J \setminus I$	{3,4,5},{2,4,6},{2,3},{1,6},{1,5},∅		
2	I	{1,2},{1,3},{1,4},{2,5},{3,5,6},{4,5,6}	9	
	$J \setminus I$	{3,4,5},{2,4,6},{2,3},{1,6},∅,∅		
3	I	{1,2},{1,3},{1,4},{2,5,6},{3,5,6},{4,5,6}	10	
	$J \setminus I$	{3,4,5},{2,4},{2,3,6},∅,∅,∅		
4	I	{1,2},{1,3},{2,4},{3,4,5},{1,5,6},{4,5,6}	10	
	$J \setminus I$	{3,4,6},{2,6},{1,5},∅,∅,∅		
5	I	{1,2},{1,3},{1,4,5},{2,4,6},{3,4,6},{4,5,6}	11	
	$J \setminus I$	{3,5},{2,5,6},∅,∅,∅,∅		
6	I	{1,2},{1,3},{2,4,5},{3,4,5},{2,4,6},{3,5,6}	11	
	$J \setminus I$	{3,4},{2,5,6},∅,∅,∅,{1}		
7	I	{1,5},{2,6},{1,2,3},{2,3,4},{3,4,5},{4,5,6}		
7.1	$J \setminus I$	{2,4},{1,3},{6},∅,∅,∅	11	
7.2	$J \setminus I$	{2,4,6},{1,3,5},{6},∅,∅,{1}	10	7.1
8	I	{1,2},{1,3,4},{1,3,5},{2,4,6},{3,4,6},{2,5,6}		
8.1	$J \setminus I$	{3,6},{5},{4},∅,∅,∅	12	
8.2	$J \setminus I$	{3,6},{5},{4},{5},∅,{4}	11	8.1
8.3	$J \setminus I$	{3,5,6},{5},{2,4},∅,∅,{1}	11	8.1
8.4	$J \setminus I$	{3,5,6},{5},{2,4},{5},∅,{1,4}	10	8.2,8.3
9	I	{1,2},{1,3,4},{1,3,5},{2,4,6},{3,4,6},{4,5,6}		
9.1	$J \setminus I$	{3,6},∅,∅,{5},∅,{2}	12	
9.2	$J \setminus I$	{3,5,6},∅,{2},∅,∅,∅	12	
10	I	{1,2},{1,3,4},{1,3,5},{2,4,6},{3,5,6},{4,5,6}	12	
	$J \setminus I$	{3,6},∅,∅,{5},∅,{2}		
11	I	{1,2,3},{1,2,4},{1,2,5},{1,3,6},{2,4,6},{3,4,6}	12	
	$J \setminus I$	{5},{5},{3,4},∅,∅,{5}		
12	I	{1,2,3},{1,2,4},{1,2,5},{1,3,6},{2,4,6},{3,5,6}	13	
	$J \setminus I$	∅,{5},{4},∅,∅,{4}		
13	I	{1,2,3},{2,3,4},{3,4,5},{4,5,6},{1,5,6},{1,2,6}		
13.1	$J \setminus I$	{4},{1,5},{2,6},{3},∅,∅	12	
13.2	$J \setminus I$	{4},{1},{6},{3},{2},{5}	12	
13.3	$J \setminus I$	{4,6},{1,5},{2,6},{3},∅,{3}	11	13.1
13.4	$J \setminus I$	{4},{1,5},{2,6},{3},{2},{5}	11	13.1,13.2
13.5	$J \setminus I$	{4,6},{1,5},{2,6},{3},{2},{3,5}	10	13.3,13.4
13.6	$J \setminus I$	{4,6},{1,5},{2,6},{1,3},{2,4},{3,5}	9	13.5
14	I	{1,2},{1,3},{1,4},{2,5},{4,5},{3,6},{5,6}	6	
	$J \setminus I$	{3,4,5},{2,4,6},{2,3,5},{1,4,6},{1,2,6},{1,5},{2,3,4}		
15	I	{1,2},{1,3,4},{1,3,5},{1,4,6},{2,5,6},{3,5,6},{4,5,6}		
15.1	$J \setminus I$	{3,4},{2},∅,∅,∅,∅,∅	12	
15.2	$J \setminus I$	{3,4,6},{2},∅,{2},{4},∅,{2}	11	15.1
16	I	{1,2,3},{1,2,4},{1,2,5},{1,3,6},{2,4,6},{3,4,6},{3,5,6}		
16.1	$J \setminus I$	∅,∅,∅,∅,∅,{5},{4}	13	
16.2	$J \setminus I$	{5},∅,{3},{5},∅,{5},{1,4}	12	16.1
16.3	$J \setminus I$	∅,{5},{4},∅,∅,{5},{4}	12	16.1
16.4	$J \setminus I$	{5},{5},{3,4},{5},∅,{5},{1,4}	11	16.2,16.3
17	I	{1,2,3},{1,2,4},{1,2,5},{1,3,6},{2,4,6},{3,5,6},{4,5,6}	13	
	$J \setminus I$	∅,∅,∅,∅,∅,{4},{3}		
18	I	{1,2,3},{2,3,4},{3,4,5},{1,4,5},{1,2,5},{3,4,6},{1,4,6},{1,2,6}	12	
	$J \setminus I$	∅,∅,{6},{6},{6},{5},{5},{5}		
19	I	{3,4,5},{1,4,5},{1,2,5},{1,2,3},{1,5,6},{2,3,4,6}		
19.1	$J \setminus I$	∅,{6},∅,∅,{4},∅	14	
19.2	$J \setminus I$	∅,{6},{6},∅,{2,4},∅	13	19.1

Table 1: Extended minimal support sets $\mathcal{E} = \{(I_\alpha, J_\alpha)\}_{\alpha=1,\dots,m}$ and dimensions of components of exceptional extreme matrices in $\mathcal{COP}^6$. Since $I_\alpha \subset J_\alpha$, for brevity only I and $J \setminus I$ are given for each minimal zero. The last column lists which components lie in the closure of other components by their construction in [1].

No.	may possibly be in the closure of
O5.2	8.1,16.1,17
1	2,3,4,5,6,7.1,8.1,9.1,9.2,10,15.1,16.1,17
2	3,4,5,6,7.1,8.1,9.1,9.2,10,15.1,16.1,17
3	5,9.2,15.1,16.1,17
4	5,6,7.1,8.1,9.1,9.2,10,15.1,16.1,17
5	9.2,15.1,16.1,17
6	8.1,9.1,9.2,10,16.1,17
7.1	8.1,9.1,10,15.1,16.1,17
8.1	16.1
9.1	16.1
9.2	16.1,17
10	17
11	19.1
12	19.1
13.1	
13.2	
14	1,2,3,4,5,6,7.1,8.1,9.1,9.2,10,11,12,13.1,13.2,15.1,16.1,17,18
15.1	16.1,17
16.1	
17	
18	12,16.1,19.1
19.1	

Table 2: Pairs of components of exceptional extremal matrices in $\mathcal{COP}^6$ satisfying both the criterion in Lemma 2.3 and a strict inequality on the dimensions.